

Higher Dimensional Homology Algebra IV:Projective Resolutions and Derived 2-Functors in $(\mathcal{R}\text{-}2\text{-Mod})$

Fang HUANG, Shao-Han CHEN, Wei CHEN, Zhu-Jun ZHENG*

Abstract: In this paper, we will construct the projective resolution of any \mathcal{R} -2-module, define the derived 2-functor and give some related properties of the derived 2-functor.

Keywords: \mathcal{R} -2-Module; Projective Resolution; Derived 2-Functor

1 Introduction

A 2-ring \mathcal{R} is a category with categorical ringed structure(see [8]). As 1-dimensional algebra, we defined \mathcal{R} -2-modules [5] in a different way with M.Dupont's 2-modules in his PhD. thesis[2]. An \mathcal{R} -2-module we mentioned in this paper is $(\mathcal{A}, I, \cdot, a, b, i, z)$, where \mathcal{A} is a symmetric 2-group with \mathcal{R} -2-module structure \cdot , I is the unit object under \cdot , a, b, i, z are natural isomorphisms satisfying canonical properties [5].

Based on the works of A.del Río, J. Martínez-Moreno and E. M. Vitale[3], we defined the left derived 2-functor in the 2-category (2-SGp) and gave a fundamental property of the derived 2-functor in our third paper [7] of the series of higher dimensional homology algebra. In [2, 5], the authors showed that

*Supported in part by NSFC with grant Number 10971071 and Provincial Foundation of Innovative Scholars of Henan.

the 2-category $(\mathcal{R}\text{-2-Mod})$ is an abelian 2-category which has enough projective(injective) objects([6, 10]). Naturally, we will consider the higher dimensional homological theory in $(\mathcal{R}\text{-2-Mod})$.

The aim of this paper is to develop a homological theory in the 2-category $(\mathcal{R}\text{-2-Mod})$ just like the 1-dimensional case. We will construct the projective resolution of any \mathcal{R} -2-module, which is unique up to 2-chain homotopy(Definition 3) and give the definition of the left derived 2-functor in $(\mathcal{R}\text{-2-Mod})$. Moreover, we shall give a fundamental property of the derived 2-functor. In our paper, most results are similar to [7], just replacing the morphisms of symmetric 2-groups by morphisms of \mathcal{R} -2-modules. The most different and difficult are to give the \mathcal{R} -2-module structures of relative kernel and cokernel.

The present paper is organized as follows. In section 2, we give some basic facts on \mathcal{R} -2-modules such as the relative kernel and cokernel which are appeared in [2, 3, ?] for symmetric 2-group case. The homology \mathcal{R} -2-modules of a complex of \mathcal{R} -2-modules appear in this section, too. In the last section, we mainly give the definition of projective resolution of an \mathcal{R} -2-module and give its construction(Proposition 2). After the basic definition of derived 2-functors from abelian 2-category $(\mathcal{R}\text{-2-Mod})$ to $(\mathcal{S}\text{-2-Mod})$ ([2, 5]), we obtain our main result(Theorem 2).

This is the fourth paper of the series works on higher dimensional homological algebra.

2 Preliminary

In this section, we give the definitions and constructions of the relative (co)kernel in $(\mathcal{R}\text{-2-Mod})$ from the definitions of them given in [2, 3], and then give the homology \mathcal{R} -2-modules of a complex of \mathcal{R} -2-modules which is similar to the homology symmetric 2-groups given in [7], where \mathcal{R} is a 2-ring. In this paper, we will omit the composition symbol \circ in our diagrams.

Definition 1. The relative kernel of the sequence $(F, \varphi, G) : \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $(\mathcal{R}\text{-2-Mod})$ is the triple $(Ker(F, \varphi), e_{(F, \varphi)}, \varepsilon_{(F, \varphi)})$ in $(\mathcal{R}\text{-2-Mod})$ as in the following diagram

$$\begin{array}{ccccc}
& & & 0 & \\
& & & \uparrow \varphi & \searrow \\
Ker(F, \varphi) & \xrightarrow{e_{(F, \varphi)}} & \mathcal{A} & \xrightarrow{F} & \mathcal{B} \xrightarrow{G} \mathcal{C} \\
& \searrow & \downarrow \varepsilon_{(F, \varphi)} & \nearrow & \\
& & 0 & &
\end{array}$$

with $\varepsilon_{(F, \varphi)}$ compatible with φ , i.e. the following diagram commutes

$$\begin{array}{ccc}
GFe_{(F, \varphi)} & \xrightarrow{\varphi e_{(F, \varphi)}} & 0e_{(F, \varphi)} \\
\downarrow G\varepsilon_{(F, \varphi)} & & \downarrow can \\
G0 & \xrightarrow{can} & 0
\end{array}$$

and satisfies the following universal property:

Given a diagram in $(\mathcal{R}\text{-2-Mod})$

$$\begin{array}{ccccc}
& & & 0 & \\
& & & \uparrow \varphi & \searrow \\
\mathcal{K} & \xrightarrow{E} & \mathcal{A} & \xrightarrow{F} & \mathcal{B} \xrightarrow{G} \mathcal{C} \\
& \searrow & \downarrow \psi & \nearrow & \\
& & 0 & &
\end{array}$$

with ψ compatible with φ , there is a factorization

$$(E' : \mathcal{K} \rightarrow Ker(F, \varphi), \psi' : e_{(F, \varphi)} \circ E' \Rightarrow E)$$

in $(\mathcal{R}\text{-2-Mod})$ through $(e_{(F, \varphi)}, \varepsilon_{(F, \varphi)})$, that is the following diagram commutes

$$\begin{array}{ccc}
Fe_{(F, \varphi)}E & \xrightarrow{\varepsilon_{(F, \varphi)} \circ E'} & 0E' \\
\downarrow F\psi & & \downarrow can \\
FE & \xrightarrow{can} & 0
\end{array}$$

and if (E'', ψ'') is another factorization of (E, ψ) through $(e_{(F, \varphi)}, \varepsilon_{(F, \varphi)})$, then there is a unique 2-morphism $e : E' \Rightarrow E''$, such that

$$\begin{array}{ccc}
e_{(F,\varphi)}E' & \xrightarrow{e_{(F,\varphi)}e} & e_{(F,\varphi)}E'' \\
\searrow \psi & & \swarrow \psi' \\
& E &
\end{array}$$

commutes.

The existence of relative kernel is given similarly to the general kernel[5].

First, $Ker(F, \varphi)$ is a symmetric 2-group(see [3, 7]) consisting of:

- An object is a pair $(A \in obj(\mathcal{A}), a : F(A) \rightarrow 0)$ such that the following diagram commutes

$$\begin{array}{ccc}
G(F(A)) & \xrightarrow{G(a)} & G(0) \\
\searrow \varphi_A & & \swarrow \simeq \\
& 0 &
\end{array}$$

- A morphism $f : (A, a) \rightarrow (A', a')$ is a morphism $f : A \rightarrow A'$ of \mathcal{A} such that the following diagram commutes

$$\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
\searrow a & & \swarrow a' \\
& 0 &
\end{array}$$

Second, $Ker(F, \varphi)$ is an \mathcal{R} -2-module:

There is a bifunctor

$$\begin{aligned}
\cdot : \mathcal{R} \times Ker(F, \varphi) &\rightarrow Ker(F, \varphi) \\
(r, (A, a)) &\mapsto r \cdot (A, a) \triangleq (r \cdot A, r \cdot a), \\
(x, f) &\mapsto x \cdot f,
\end{aligned}$$

where $r \cdot A$ and $x \cdot f$ are the object and morphism in \mathcal{A} under its \mathcal{R} -2-module structure, respectively, $r \cdot a$ is the composition morphism $F(r \cdot A) \simeq r \cdot F(A) \xrightarrow{r \cdot a} r \cdot 0 \simeq 0$. The above bifunctor is well-defined. In fact, for $(A, a) \in obj(Ker(F, \varphi))$ with $G(a) = \varphi_A$, there is $G(r \cdot a) = r \cdot G(a) = r \cdot \varphi_A = \varphi_{r \cdot A}$ from the basic properties

of \mathcal{R} -2-modules. Moreover, the natural isomorphisms in the definition of \mathcal{R} -2-modules and the universal property are given as general kernels (more details see [5]).

Definition 2. The relative cokernel of the sequence $(F, \varphi, G) : \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $(\mathcal{R}\text{-2-Mod})$ is the triple $(Coker(\varphi, G), p_{(\varphi, G)}, \pi_{(\varphi, G)})$ in $(\mathcal{R}\text{-2-Mod})$ as in the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & \curvearrowright & \uparrow \varphi & \searrow & & & \\
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{p_{(\varphi, G)}} & Coker(\varphi, G) \\
 & & \downarrow \pi_{(\varphi, G)} & \searrow & & \nearrow & \\
 & & 0 & & & &
 \end{array}$$

with $\pi_{(\varphi, G)}$ compatible with φ , i.e. the following diagram commutes

$$\begin{array}{ccc}
 p_{(\varphi, G)} GF & \xrightarrow{p_{(\varphi, G)} \varphi} & p_{(\varphi, G)} 0 \\
 \pi_{(\varphi, G)} F \downarrow & & \downarrow \text{can} \\
 0F & \xrightarrow{\text{can}} & 0
 \end{array}$$

and satisfies the following universal property:

Given a diagram in $(\mathcal{R}\text{-2-Mod})$

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & \curvearrowright & \uparrow \varphi & \searrow & & & \\
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{E} & \mathcal{K} \\
 & & \downarrow \psi & \searrow & & \nearrow & \\
 & & 0 & & & &
 \end{array}$$

with ψ compatible with φ , there is a factorization

$$(E' : Coker(\varphi, G) \rightarrow \mathcal{K}, \psi' : E' \circ p_{(\varphi, G)} \Rightarrow E)$$

in $(\mathcal{R}\text{-2-Mod})$ through $(p_{(\varphi, G)}, \pi_{(\varphi, G)})$, that is the following diagram commutes

$$\begin{array}{ccc}
E' p_{(\varphi, G)} G & \xrightarrow{E' \pi_{(\varphi, G)}} & E' 0 \\
\downarrow \psi \cdot G & & \downarrow can \\
EG & \xrightarrow{\psi} & 0
\end{array}$$

and if (E'', ψ'') is another factorization of (E, ψ) through $(p_{(\varphi, G)}, \pi_{(\varphi, G)})$, then there is a unique 2-morphism $e : E' \Rightarrow E''$, such that

$$\begin{array}{ccc}
E' p_{(\varphi, G)} & \xrightarrow{\eta_{(\varphi, G)}} & E' p_{(\varphi, G)} \\
\downarrow \psi \cdot & & \downarrow \psi \cdot \\
& E &
\end{array}$$

commutes.

The existence of relative cokernel is also similar to the general cokernel[5].

First, $Coker(\varphi, G)$ is a symmetric 2-group(see [3, 7]) consisting of:

- Objects are those of \mathcal{C} .
- A morphism from X to Y is an equivalent class of a pair $(B, f) : X \rightarrow Y$ with $B \in obj(\mathcal{B})$ and $f : X \rightarrow G(B) + Y$. For two morphisms $(B, f), (B', f') : X \rightarrow Y$ are equivalent if there is $A \in obj(\mathcal{A})$ and $a : B \rightarrow F(A) + B'$ such that the following diagram commutes

$$\begin{array}{ccc}
X & \xrightarrow{f} & G(B) + Y \\
f' \downarrow & & \downarrow G(a) + 1 \\
G(B') + Y & & G(F(A) + B') + Y \\
\cong \downarrow & & \downarrow \cong \\
0 + G(B') + Y & \xleftarrow{\varphi_A + 1 + 1} & GF(A) + G(B') + Y
\end{array}$$

Second, $Coker(\varphi, G)$ is an \mathcal{R} -2-module.

There is a bifunctor

$$\begin{aligned} \cdot : \mathcal{R} \times \text{Coker}(\varphi, G) &\rightarrow \text{Coker}(\varphi, G) \\ (r, X) &\mapsto r \cdot X, \\ (r_1 \xrightarrow{x} r_2, X \xrightarrow{[B, f]} Y) &\mapsto r_1 \cdot X \xrightarrow{[r_1 \cdot B, \bar{f}]} r_2 \cdot Y, \end{aligned}$$

where $r \cdot X$ and $r_1 \cdot B$ are the objects in \mathcal{C} and \mathcal{B} under the \mathcal{R} -2-module structures of them, respectively, \bar{f} is the composition morphism $r_1 \cdot X \xrightarrow{r_1 \cdot f} r_1 \cdot (G(B) + Y) \simeq r_1 \cdot G(B) + r_1 \cdot Y \xrightarrow{1+x \cdot Y} G(r_1 \cdot B) + r_2 \cdot Y$. This bifunctor is well-defined. In fact, if $[B, f] = [B', f'] : X \rightarrow Y$, i.e. there exist $A \in \text{obj}(\mathcal{A})$ and $a : B \rightarrow F(A) + B'$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & G(B) + Y \\ f' \downarrow & & \downarrow G(a) + 1 \\ G(B') + Y & & G(F(A) + B') + Y \\ \simeq \downarrow & & \downarrow \simeq \\ 0 + G(B') + Y & \xleftarrow{\varphi_A + 1 + 1} & GF(A) + G(B') + Y \end{array}$$

Hence, there exist $r_1 \cdot A \in \text{obj}(\mathcal{A})$ and $r_1 \cdot a : r_1 \cdot B \rightarrow F(r_1 \cdot A) + r_1 \cdot B'$ such that the following diagram commutes

$$\begin{array}{c} \begin{array}{c} \xrightarrow{\quad \bar{f} \quad} \\ \begin{array}{c} r_1 \cdot X \xrightarrow{r_1 \cdot f} r_1 \cdot (G(B) + Y) \xrightarrow{\simeq} r_1 \cdot G(B) + r_1 \cdot Y \xrightarrow{1+x \cdot Y} r_1 \cdot G(B) + r_2 \cdot Y \xrightarrow{\simeq} G(r_1 \cdot B) + r_2 \cdot Y \\ \downarrow r_1 \cdot f' \quad \downarrow r_1 \cdot (G(a) + 1) \quad \downarrow r_1 \cdot G(a) + 1 \quad \downarrow r_1 \cdot G(a) + 1 \quad \downarrow G(r_1 \cdot a) + 1 \\ r_1 \cdot (G(B') + Y) \quad r_1 \cdot (G(F(A) + B') + Y) \quad r_1 \cdot G(F(A) + B') + r_1 \cdot Y \quad r_1 \cdot G(F(A) + B') + r_2 \cdot Y \quad G(F(r_1 \cdot A) + r_1 \cdot B') + r_2 \cdot Y \\ \downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq \\ r_1 \cdot (GF(A) + G(B') + Y) \quad r_1 \cdot (GF(A) + G(B')) + r_2 \cdot Y \quad GF(r_1 \cdot A) + G(r_1 \cdot B') + r_2 \cdot Y \\ \downarrow r_1 \cdot (\varphi_A + 1 + 1) \quad \downarrow r_1 \cdot (\varphi_A + 1 + 1) \quad \downarrow \varphi_{r_1 \cdot A} + 1 + 1 \\ r_1 \cdot G(B') + r_1 \cdot Y \quad r_1 \cdot (0 + G(B') + Y) \quad r_1 \cdot (0 + G(B')) + r_2 \cdot Y \quad 0 + G(r_1 \cdot B') + r_2 \cdot Y \\ \downarrow 1+x \cdot Y \quad \downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq \\ r_1 \cdot G(B') + r_2 \cdot Y \quad r_1 \cdot (0 + G(B')) + r_1 \cdot Y \quad r_1 \cdot (0 + G(B')) + r_2 \cdot Y \quad 0 + G(r_1 \cdot B') + r_2 \cdot Y \\ \downarrow \bar{f}' \quad \downarrow \simeq \quad \downarrow \simeq \quad \downarrow \simeq \\ r_1 \cdot G(B') + r_2 \cdot Y \quad r_1 \cdot (0 + G(B')) + r_1 \cdot Y \quad r_1 \cdot (0 + G(B')) + r_2 \cdot Y \quad 0 + G(r_1 \cdot B') + r_2 \cdot Y \end{array} \end{array} \end{array}$$

Then $x \cdot [B, f] = [r_1 \cdot B, \bar{f}] = [r_1 \cdot B', \bar{f}'] = r_1 \cdot [B', f'] : r_1 \cdot X \rightarrow r_2 \cdot Y$.

Moreover, the natural isomorphisms in the definitions of \mathcal{R} -2-module and the universal property are given as general cokernel (more details see [5]).

Remark 1. Just like symmetric 2-group discussed in [1, 2, 3, ?, 7], we can give the definitions of (relative-)2-exact, cohomology \mathcal{R} -2-modules in $(\mathcal{R}\text{-2-Mod})$.

A complex of \mathcal{R} -2-modules in $(\mathcal{R}\text{-2-Mod})$ is a sequence

$$\mathcal{A}. = \cdots \xrightarrow{L_{n+1}} \mathcal{A}_n \xrightarrow{L_n} \mathcal{A}_{n-1} \xrightarrow{L_{n-1}} \mathcal{A}_{n-2} \xrightarrow{L_{n-2}} \cdots \xrightarrow{L_2} \mathcal{A}_1 \xrightarrow{L_1} \mathcal{A}_0$$

together with a family of 2-morphisms $\{\alpha_n : L_{n-1} \circ L_n \Rightarrow 0\}_{n \geq 2}$ in $(\mathcal{R}\text{-2-Mod})$ such that, for all n , the following diagram commutes

$$\begin{array}{ccc} L_{n-1}L_nL_{n+1} & \xrightarrow{\alpha_n L_{n+1}} & 0L_{n+1} \\ \downarrow L_{n-1}\alpha_{n+1} & & \downarrow \text{can} \\ L_{n-1}0 & \xrightarrow{\text{can}} & 0 \end{array}$$

Consider part of the complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \nearrow & \alpha_{n+2} \Uparrow & \searrow & \Uparrow \alpha_n & \searrow & \\ \mathcal{A}_{n+2} & \xrightarrow{L_{n+2}} & \mathcal{A}_{n+1} & \xrightarrow{L_{n+1}} & \mathcal{A}_n & \xrightarrow{L_n} & \mathcal{A}_{n-1} & \xrightarrow{L_{n-1}} & \mathcal{A}_{n-2} \\ & & & \searrow & \alpha_{n+1} \Downarrow & \nearrow & & & \\ & & & & 0 & & & & \end{array}$$

Based on the universal properties of relative kernel $\text{Ker}(L_n, \alpha_n)$, we have the following diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \nearrow & \alpha_{n+2} \Uparrow & \searrow & \Uparrow \alpha_n & \searrow & \\ \mathcal{A}_{n+2} & \xrightarrow{L_{n+2}} & \mathcal{A}_{n+1} & \xrightarrow{L_{n+1}} & \mathcal{A}_n & \xrightarrow{L_n} & \mathcal{A}_{n-1} & \xrightarrow{L_{n-1}} & \mathcal{A}_{n-2} \\ & \searrow & \downarrow \overline{\alpha_{n+2}} & \searrow L_{n+1}' & \Uparrow \alpha_{n+1}' & \uparrow e_{(L_n, \alpha_n)} & & & \\ & & & & \text{Ker}(L_n, \alpha_n) & & & & \end{array}$$

Similarly as the definition of (co)homology 2-group in [3, 7], the n th homology \mathcal{R} -2-module $\mathcal{H}_n(\mathcal{A}.)$ of the complex $\mathcal{A}.$ defined as the relative cokernel $\text{Coker}(\overline{\alpha_{n+2}}, L_{n+1}')$.

Note that, to get $\mathcal{H}_0(\mathcal{A})$ and $\mathcal{H}_1(\mathcal{A})$, we have to complete the complex \mathcal{A} on the right with the two zero morphisms and two canonical 2-morphisms $\cdots \xrightarrow{L_2} \mathcal{A}_1 \xrightarrow{L_1} \mathcal{A}_0 \xrightarrow{0} 0 \xrightarrow{0} 0$, $\text{can} : 0 \circ L_1 \Rightarrow 0$, $\text{can} : 0 \circ 0 \Rightarrow 0$.

The explicit description of $\mathcal{H}_n(\mathcal{A})$ can also be given from the existence of relative kernel and relative cokernel in $(\mathcal{R}\text{-2-Mod})$ like the symmetric 2-group case in [3, 7].

A morphism $(F, \lambda) : \mathcal{A} \rightarrow \mathcal{B}$ of complexes in $(\mathcal{R}\text{-2-Mod})$ is a picture in the following diagram such that the following diagram commutes

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \uparrow \alpha_{n+1} & & & \\
\cdots & \xrightarrow{L_{n+2}} & \mathcal{A}_{n+1} & \xrightarrow{L_{n+1}} & \mathcal{A}_n & \xrightarrow{L_n} & \mathcal{A}_{n-1} \xrightarrow{L_{n-1}} \mathcal{A}_{n-2} \cdots \\
& & \downarrow F_{n+1} & \swarrow \lambda_{n+1} & \downarrow F_n & \swarrow \lambda_n & \downarrow F_{n-1} \swarrow \lambda_{n-1} \downarrow F_{n-2} \\
& & \mathcal{B}_{n+1} & \xrightarrow{M_{n+1}} & \mathcal{B}_n & \xrightarrow{M_n} & \mathcal{B}_{n-1} \xrightarrow{M_{n-1}} \mathcal{B}_{n-2} \cdots \\
& & & \downarrow \beta_{n+1} & & & \\
& & & 0 & & &
\end{array}$$

where $F_n : \mathcal{A}_n \rightarrow \mathcal{B}_n$ is 1-morphism in $(\mathcal{R}\text{-2-Mod})$, $\lambda_n : F_{n-1} \circ L_n \Rightarrow M_n \circ F_n$ is 2-morphism in $(\mathcal{R}\text{-2-Mod})$, for each n , making the following diagram commutative

$$\begin{array}{ccccc}
F_{n-1} L_n L_{n+1} & \xrightarrow{\lambda_n L_{n+1}} & M_n F_n L_{n+1} & \xrightarrow{M_n \lambda_{n+1}} & M_n M_{n+1} F_{n+1} \\
\downarrow F_{n-1} \alpha_{n+1} & & & & \downarrow \beta_{n+1} F_{n+1} \\
F_{n-1} 0 & \xrightarrow{\text{can}} & 0 & \xleftarrow{\text{can}} & 0 F_{n+1}
\end{array}$$

Such a morphism induces, for each n , a morphism of homology \mathcal{R} -2-modules $\mathcal{H}_n(F) : \mathcal{H}_n(\mathcal{A}) \rightarrow \mathcal{H}_n(\mathcal{B})$ from the universal properties of relative kernels and cokernels. It can be described as follows:(more details see [3, 7]).

Given an object $(A_n \in \text{obj}(\mathcal{A}_n), a_n : L_n(A_n) \rightarrow 0)$ of $\mathcal{H}_n(\mathcal{A})$, we have $\mathcal{H}_n(F)(A_n, a_n) = (F_n(A_n) \in \text{obj}(\mathcal{B}_n), b_n : M_n(F_n(A_n)) \rightarrow 0)$, where b_n is the composition $M_n(F_n(A_n)) \xrightarrow{(\lambda_n)_{A_n}^{-1}} F_{n-1} L_n(A_n) \xrightarrow{F_{n-1}(a_n)} F_{n-1}(0) \simeq 0$.

Given a morphism $[X_{n+1} \in \text{obj}(\mathcal{A}_{n+1}), x_{n+1} : A_n \rightarrow L_{n-1}(X_{n+1}) + A'_n] : (A_n, a_n) \rightarrow (A'_n, a'_n)$ in $\mathcal{H}_n(\mathcal{A})$, we have $\mathcal{H}_n(F)[X_{n+1}, x_{n+1}] = [F_{n+1}(X_{n+1}) \in$

$obj(\mathcal{B}_{n+1}), \overline{x_{n+1}} : F_n(A_n) \rightarrow M_{n+1}(F_{n+1}(X_{n+1}) + F_n(A'_n)) : (F_n(A_n, b_n) \rightarrow (F_n(A'_n), b'_n),$
where $\overline{x_{n+1}}$ is the composition $F_n(A_n) \xrightarrow{F_n(x_{n+1})} F_n(L_{n+1}(X_{n+1} + A'_n)) \simeq F_n L_{n+1}(X_{n+1}) + F_n(A'_n) \xrightarrow{(\lambda_{n+1})_{X_{n+1}} + 1} M_{n+1}(F_{n+1}(X_{n+1})) + F_n(A'_n).$

Remark 2. 1. For a complex of \mathcal{R} -2-modules which is relative 2-exact in each point, the (co)homology \mathcal{R} -2-modules are always zero \mathcal{R} -2-module (only one object and one morphism).

2. For morphisms $\mathcal{A} \xrightarrow{(F, \lambda)} \mathcal{B} \xrightarrow{(G, \mu)} \mathcal{C}$ of complexes in $(\mathcal{R}\text{-}2\text{-Mod})$, their composition is given by $(G_n \circ F_n, (\mu_n \circ F_{n+1}) \star (G_n \circ \lambda_n))$, for $n \in \mathbb{Z}$, where \star is the vertical composition of 2-morphisms in 2-category([9]). Moreover, $\mathcal{H}_n(G \circ F) \simeq \mathcal{H}_n(G) \circ \mathcal{H}_n(F).$

Definition 3. Let $(F, \lambda), (G, \mu) : (\mathcal{A}, L, \alpha) \rightarrow (\mathcal{B}, M, \beta)$ be two morphisms of complexes of \mathcal{R} -2-modules. If there is a family of 1-morphisms $\{H_n : \mathcal{A}_n \rightarrow \mathcal{B}_{n+1}\}_{n \in \mathbb{Z}}$ and a family of 2-morphisms $\{\tau_n : F_n \Rightarrow M_{n+1} \circ H_n + H_{n-1} \circ L_n + G_n : \mathcal{A}_n \rightarrow \mathcal{B}_n\}_{n \in \mathbb{Z}}$ satisfying the obvious compatible conditions, i.e. the following diagram commutes

$$\begin{array}{ccc}
F_{n-1}L_n & \xRightarrow{\lambda_n} & M_n F_n \\
\tau_{n-1} L_n \downarrow & & \downarrow M_n \tau_n \\
(M_n H_{n-1} + H_{n-2} L_{n-1} + G_{n-1}) L_n & & M_n (M_{n+1} H_n + H_{n-1} L_n + G_n) \\
\text{can} \downarrow & & \downarrow \text{can} \\
M_n H_{n-1} L_n + H_{n-2} L_{n-1} L_n + G_{n-1} L_n & & M_n M_{n+1} H_n + M_n H_{n-1} L_n + M_n G_n \\
1 + H_{n-2} \alpha_n + 1 \downarrow & & \downarrow \beta_{n+1} H_n + 1 + 1 \\
M_n H_{n-1} L_n + H_{n-2} (0) + G_{n-1} L_n & & 0 + M_n H_{n-1} L_n + M_n G_n \\
\text{can} \downarrow & & \downarrow 1 + \mu_n^{-1} \\
M_n H_{n-1} L_n + 0 + G_{n-1} L_n & \xRightarrow{\text{can}} & M_n H_{n-1} L_n + G_{n-1} L_n
\end{array}$$

We call the above morphisms $(F, \lambda), (G, \mu)$ are 2-chain homotopy in $(\mathcal{R}\text{-}2\text{-Mod})$.

Like the symmetric 2-group case, we have

Proposition 1. Let $(F, \lambda), (G, \mu) : (\mathcal{A}, L, \alpha) \rightarrow (\mathcal{B}, M, \beta)$ be two morphisms of complexes of \mathcal{R} -2-modules. If they are 2-chain homotopy, there is an equivalence $\mathcal{H}_n(F) \simeq \mathcal{H}_n(G)$ between induced morphisms.

Lemma 1. *Let \mathcal{R}, \mathcal{S} be 2-rings, $(F., \lambda.), (G., \mu.) : (\mathcal{A}, L., \alpha.) \rightarrow (\mathcal{B}, M., \beta.)$ be two 2-chain homotopy morphisms of complexes of \mathcal{R} -2-modules and $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ be a 2-functor. Then $T(F., \lambda.)$ is 2-chain homotopic to $T(G., \mu.)$ in $(\mathcal{S}\text{-2-Mod})$.*

3 Projective Resolution and Derived 2-Functor in $(\mathcal{R}\text{-2-Mod})$

In this section we will construct a projective resolution of any \mathcal{R} -2-module, define the left derived 2-functor and then give the basic property of this derived 2-functor.

Definition 4. Let \mathcal{M} be an \mathcal{R} -2-module. A projective resolution of \mathcal{M} in $(\mathcal{R}\text{-2-Mod})$ is a complex of \mathcal{R} -2-modules which is relative 2-exact in each point as in the following diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \nearrow & & \nearrow & \\
 \dots & \xrightarrow{F_3} & \mathcal{P}_2 & \xrightarrow{F_2} & \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_0 \xrightarrow{F_0} \mathcal{M} \xrightarrow{0} 0 \\
 & & \nwarrow & \alpha_2 \Uparrow & \nwarrow & \nwarrow & \nwarrow \\
 & & & 0 & & 0 & \\
 & & & \searrow & & \searrow & \\
 & & & 0 & & 0 &
 \end{array}$$

with $\mathcal{P}_n (n \geq 0)$ projective objects in $(\mathcal{R}\text{-2-Mod})$. i.e. the above complex is relative 2-exact in each \mathcal{P}_i and \mathcal{M} .

Proposition 2. *Every \mathcal{R} -2-module \mathcal{M} has a projective resolution in $(\mathcal{R}\text{-2-Mod})$.*

Sketch of proof. The construction of projective resolution of \mathcal{M} is similar to symmetric 2-group case.

For \mathcal{M} , there is an essentially surjective morphism $F_0 : \mathcal{P}_0 \rightarrow \mathcal{M}$, with \mathcal{P}_0 projective object in $(\mathcal{R}\text{-2-Mod})$ ([6, 10]). Then we get a sequence as follows

$$\begin{array}{ccc}
 & 0 & \\
 & \nearrow & \\
 \mathcal{P}_0 & \xrightarrow{F_0} & \mathcal{M} \xrightarrow{0} 0
 \end{array}
 \quad \text{S.1.}$$

where $0 : \mathcal{M} \rightarrow 0$ is the zero morphism[5] in $(\mathcal{R}\text{-2-Mod})$, 0 is the \mathcal{R} -2-module with only one object and one morphism., can is the canonical 2-morphism in $(\mathcal{R}\text{-2-Mod})$, which is given by the identity morphism of only one object of 0 .

From the existence of the relative kernel in $(\mathcal{R}\text{-2-Mod})$, we have the relative kernel $(Ker(F_0, can), e_{(F_0, can)}, \varepsilon_{(F_0, can)})$ of the sequence S.1, which is in fact the general kernel $(Ker F_0, e_{F_0}, \varepsilon_{F_0})$ [5]. For \mathcal{R} -2-module $Ker F_0$, there exists an essentially surjective morphism $G_1 : \mathcal{P}_1 \rightarrow Ker F_0$, with \mathcal{P}_1 projective object in $(\mathcal{R}\text{-2-Mod})$ ([6, 10]). Let $F_1 = e_{F_0} \circ G_1 : \mathcal{P}_1 \rightarrow \mathcal{P}_0$. Then we get the following sequence

$$\begin{array}{ccccccc}
 & & \alpha_1 \Uparrow & & & & \\
 \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_0 & \xrightarrow{F_0} & \mathcal{M} & \xrightarrow{0} & 0 \\
 & \Uparrow id & \uparrow e_{F_0} & \Downarrow \varepsilon_{F_0} & \nearrow 0 & & \\
 & G_1 & & & Ker F_0 & &
 \end{array}$$

where α_1 is the composition $F_0 \circ F_1 = F_0 \circ e_{F_0} \circ G_1 \Rightarrow 0 \circ G_1 \Rightarrow 0$ and compatible with can .

Consider the above sequence, there exists the relative kernel $(Ker(F_1, \alpha_1), e_{(F_1, \alpha_1)}, \varepsilon_{(F_1, \alpha_1)})$ in (2-SGp) . For the \mathcal{R} -2-module $Ker(F_1, \alpha_1)$, there is an essentially surjective morphism $G_2 : \mathcal{P}_2 \rightarrow Ker(F_1, \alpha_1)$, with \mathcal{P}_2 projective object in $(\mathcal{R}\text{-2-Mod})$ ([6, 10]). Let $F_2 = e_{(F_1, \alpha_1)} \circ G_2 : \mathcal{P}_2 \rightarrow \mathcal{P}_1$. Then we get a sequence

$$\begin{array}{ccccccc}
 & & Ker(F_1, \alpha_1) & & & & \\
 & \nearrow G_2 & \downarrow e_{(F_1, \alpha_1)} & \searrow 0 & & & \\
 \mathcal{P}_2 & \xrightarrow{F_2} & \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_0 & \xrightarrow{F_0} & \mathcal{M} \xrightarrow{0} 0 \\
 & \searrow \alpha_2 & \downarrow \alpha_1 & \nearrow \alpha_1 & \nearrow 0 & & \\
 & 0 & & 0 & & &
 \end{array}$$

where α_2 is the composition $F_1 \circ F_2 = F_1 \circ e_{(F_1, \alpha_1)} \circ G_2 \Rightarrow 0 \circ G_2 \Rightarrow 0$ and compatible with α_1 .

Using the same method, we get a complex of \mathcal{R} -2-modules

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& \nearrow & \alpha_2 \Uparrow & \searrow & \nearrow & \Uparrow_{can} & \searrow \\
\cdots & \xrightarrow{F_3} & \mathcal{P}_2 & \xrightarrow{F_2} & \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_0 \xrightarrow{F_0} \mathcal{M} \xrightarrow{0} 0 \\
& & & & \searrow & \alpha_1 \Downarrow & \nearrow \\
& & & & 0 & &
\end{array}$$

The proof of relative 2-exactness of the sequence is the same as symmetric 2-group case.

Theorem 1. *Let $(F : \mathcal{P} \rightarrow \mathcal{M}, \alpha)$ be a projective resolution of \mathcal{R} -2-module \mathcal{M} , and $H : \mathcal{M} \rightarrow \mathcal{N}$ a morphism in $(\mathcal{R}\text{-2-Mod})$. Then for any projective resolution $(G : \mathcal{Q} \rightarrow \mathcal{N}, \beta)$, there is a morphism $H : \mathcal{P} \rightarrow \mathcal{Q}$ of complexes in $(\mathcal{R}\text{-2-Mod})$ together with the family of 2-morphisms $\{\varepsilon_n : G_n \circ H_n \Rightarrow H_{n-1} \circ F_n\}_{n \geq 0}$ as in the following diagram*

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& \nearrow & \alpha_2 \Uparrow & \searrow & \nearrow & \Uparrow_{can} & \searrow \\
\cdots & \xrightarrow{F_3} & \mathcal{P}_2 & \xrightarrow{F_2} & \mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_0 \xrightarrow{F_0} \mathcal{M} \xrightarrow{0} 0 \\
& \downarrow & \varepsilon_2 \Rightarrow & \downarrow & \varepsilon_1 \Rightarrow & \downarrow & \varepsilon_0 \Rightarrow \downarrow \\
& \xrightarrow{G_3} & \mathcal{Q}_2 & \xrightarrow{G_2} & \mathcal{Q}_1 & \xrightarrow{G_1} & \mathcal{Q}_0 \xrightarrow{G_0} \mathcal{N} \xrightarrow{0} 0 \\
& & \searrow & \beta_2 \Downarrow & \searrow & \beta_1 \Downarrow & \searrow \Downarrow_{can} \\
& & & 0 & & 0 & & 0
\end{array}$$

If there is another morphism between projective resolutions, they are 2-chain homotopy.

The proof of this Theorem is also similar to symmetric 2-group case in [7]. The difference is that the existence of 1-morphisms and 2-morphisms is from the properties of projective \mathcal{R} -2-modules in $(\mathcal{R}\text{-2-Mod})$.

Definition 5. Let \mathcal{R}, \mathcal{S} be 2-rings. An additive 2-functor ([2]) $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ is called right relative 2-exact if the relative 2-exactness of

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& \nearrow & \Uparrow_{can} & \searrow & \nearrow & \Uparrow_{can} & \searrow \\
0 & \xrightarrow{0} & \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \xrightarrow{0} 0 \\
& & & & \searrow & \Downarrow_{\varphi} & \nearrow \\
& & & & 0 & &
\end{array}$$

in \mathcal{A}, \mathcal{B} and \mathcal{C} implies relative 2-exactness of

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & \searrow & & \\
 T(\mathcal{A}) & \xrightarrow{T(F)} & T(\mathcal{B}) & \xrightarrow{T(G)} & T(\mathcal{C}) & \xrightarrow{0} & 0 \\
 & \searrow & \downarrow T(\varphi) & \nearrow & & & \\
 & & 0 & & & &
 \end{array}$$

in $T(\mathcal{B})$ and $T(\mathcal{C})$.

The left relative 2-exact 2-functor can be defined dually.

By Remark 2 and Proposition 1, Theorem 1, there is

Corollary 1. *Let \mathcal{R}, \mathcal{S} be 2-rings, and $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ be an additive 2-functor, and \mathcal{A} be any object of $(\mathcal{R}\text{-2-Mod})$. For two projective resolutions \mathcal{P}, \mathcal{Q} . of \mathcal{A} , there is an equivalence between homology \mathcal{S} -2-modules $\mathcal{H}_i(T(\mathcal{P}))$ and $\mathcal{H}_i(T(\mathcal{Q}))$.*

Let $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ be an additive 2-functor. There is a 2-functor

$$\begin{aligned}
 \mathcal{L}_i T : (\mathcal{R}\text{-2-Mod}) &\rightarrow (\mathcal{S}\text{-2-Mod}) \\
 \mathcal{A} &\mapsto \mathcal{L}_i T(\mathcal{A}), \\
 \mathcal{A} \xrightarrow{F} \mathcal{B} &\mapsto \mathcal{L}_i T(\mathcal{A}) \xrightarrow{\mathcal{L}_i T(F)} \mathcal{L}_i T(\mathcal{B}),
 \end{aligned}$$

where $\mathcal{L}_i T(\mathcal{A})$ is defined by $\mathcal{H}_i(T(\mathcal{P}))$, and \mathcal{P} . is the projective resolution of \mathcal{A} . $\mathcal{L}_i T$ is a well-defined 2-functor from the properties of additive 2-functor and Corollary 1.

Corollary 2. *Let $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ be a right relative 2-exact 2-functor, and \mathcal{A} be a projective object in $(\mathcal{R}\text{-2-Mod})$. Then $\mathcal{L}_i T(\mathcal{A}) = 0$ for $i \neq 0$.*

The following is a basic property of derived functors.

Theorem 2. *Let $T : (\mathcal{R}\text{-2-Mod}) \rightarrow (\mathcal{S}\text{-2-Mod})$ be a right relative 2-exact 2-functor. Then the left derived 2-functor $\mathcal{L}_* T$ takes the sequence of \mathcal{R} -2-modules*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{0} & \mathcal{C} \\
 \searrow F & \uparrow \varphi & \nearrow G \\
 & \mathcal{B} &
 \end{array}$$

which is relative 2-exact in \mathcal{A} , \mathcal{B} , \mathcal{C} to a long sequence 2-exact(similar [1, ?])in each point

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \nearrow & \Uparrow \mathcal{L}_n T(\varphi) & \searrow & \Uparrow \Psi_n & \searrow & \\
 \cdots & \longrightarrow & \mathcal{L}_n T(\mathcal{A}) & \xrightarrow{\mathcal{L}_n T(F)} & \mathcal{L}_n T(\mathcal{B}) & \xrightarrow{\mathcal{L}_n T(G)} & \mathcal{L}_n T(\mathcal{C}) & \xrightarrow{\Delta_n} & \mathcal{L}_{n-1} T(\mathcal{A}) & \xrightarrow{\mathcal{L}_{n+1} T(F)} & \mathcal{L}_{n-1} T(\mathcal{B}) & \longrightarrow \cdots \\
 & & & & \Downarrow \Sigma_n & & & & & & & \\
 & & & & 0 & & & & & & &
 \end{array}$$

In order to prove this theorem, we need the following Lemmas.

Similar to the proofs of symmetric 2-group case. We have

Lemma 2. *Let \mathcal{P} and \mathcal{Q} be projective objects in $(\mathcal{R}\text{-}2\text{-}Mod)$. Then the product category $\mathcal{P} \times \mathcal{Q}$ is a projective object in $(\mathcal{R}\text{-}2\text{-}Mod)$.*

Lemma 3. *Let $(F, \varphi, G) : \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ be an extension of $\mathcal{R}\text{-}2\text{-}modules$ in $(\mathcal{R}\text{-}2\text{-}Mod)$ (similar to [1, 3]), (\mathcal{P}, L, α) (\mathcal{Q}, N, β) be projective resolutions of \mathcal{A} and \mathcal{C} , respectively. Then there is a projective resolution $(\mathcal{K}, M, \varphi)$ of \mathcal{B} , such that $\mathcal{P} \rightarrow \mathcal{K} \rightarrow \mathcal{Q}$ forms an extension of complexes in $(\mathcal{R}\text{-}2\text{-}Mod)$.*

By the universal property of (bi)product of $\mathcal{R}\text{-}2\text{-}modules$ and the property of additive 2-functor([2, 5]). We get

Lemma 4. *Let $T : (\mathcal{R}\text{-}2\text{-}Mod) \rightarrow (\mathcal{S}\text{-}2\text{-}Mod)$ be an additive 2-functor, and \mathcal{A} , \mathcal{B} be objects in $(\mathcal{R}\text{-}2\text{-}Mod)$. Then there is an equivalence between $T(\mathcal{A} \times \mathcal{B})$ and $T(\mathcal{A}) \times T(\mathcal{B})$ in $(\mathcal{S}\text{-}2\text{-}Mod)$.*

Proof of Theorem 2. For $\mathcal{R}\text{-}2\text{-}modules$ \mathcal{A} and \mathcal{C} , choose projective resolutions $\mathcal{P} \rightarrow \mathcal{A}$ and $\mathcal{Q} \rightarrow \mathcal{C}$. By Lemma 2 and Lemma 3, there is a projective resolution $\mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{B}$ fitting into an extension $\mathcal{P} \xrightarrow{i} \mathcal{P} \times \mathcal{Q} \xrightarrow{p} \mathcal{Q}$ of projective complexes in $(2\text{-}SGp)([1])$. By Lemma 4, we obtain a complexes of extension

$$T(\mathcal{P}) \xrightarrow{T(i)} T(\mathcal{P} \times \mathcal{Q}) \xrightarrow{T(p)} T(\mathcal{Q}).$$

Similar to Theorem 4.2 in [3], the long sequence

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& \nearrow & \Uparrow \mathcal{L}_n T(\varphi) & \searrow & \Uparrow \Psi_n & \searrow & \\
\cdots & \longrightarrow & \mathcal{L}_n T(\mathcal{A}) & \xrightarrow{\mathcal{L}_n T(F)} & \mathcal{L}_n T(\mathcal{B}) & \xrightarrow{\mathcal{L}_n T(G)} & \mathcal{L}_n T(\mathcal{C}) \xrightarrow{\Delta_n} \mathcal{L}_{n-1} T(\mathcal{A}) \xrightarrow{\mathcal{L}_{n+1} T(F)} \mathcal{L}_{n-1} T(\mathcal{B}) \longrightarrow \cdots \\
& & \searrow & & \Uparrow \Sigma_n & \nearrow & \\
& & 0 & & 0 & &
\end{array}$$

is 2-exact in each point.

Acknowledgements.

We thank Prof. Zhang-Ju LIU, Prof. Yun-He SHENG for providing us the ideas of higher dimensional category theory. We also thank Prof. Ke WU and Prof. Shi-Kun WANG for useful discussions.

References

- [1] D. Bourn, E.M. Vitale, Extensions of symmetric at-groups, Homol. Homotopy Appl. 4 (2002) 103C162
- [2] M. Dupont. Abelian categories in dimension 2, PhD.Thesis. arxiv:hep-th/0809.1760v1.
- [3] A.del Río, J. Martínez-Moreno, and E. M. Vitale, Chain complexes of symmetric categorical groups, J. Pure Appl. Algebra, 196 (2005).
- [4] I.P.Freyd. Abelian categories[M], New York: Harper&Row, 1964.
- [5] F.Huang, S.H.Chen, W.Chen, Z.J.Zheng, 2-Modules and the Representation of 2-Rings. arxiv:hep-th/1005.2831 18 May 2010
- [6] F.Huang, S.H.Chen, W.Chen, Z.J.Zheng, Higher Dimensional Homology Algebra II:Projectivity. arXiv:1006.4677.
- [7] F.Huang, S.H.Chen, W.Chen, Z.J.Zheng. Higher Dimensional Homology Algebra III:Projective Resolutions and Derived 2-Functors in (2-SGp) arXiv:1006.4677.

- [8] M. Jibladze and T. Pirashvili, Third Mac Lane cohomology via categorical rings [J]. J. Homotopy Relat. Struct., 2 (2007), pp.187C216.
- [9] H.-J. Baues and M. Jibladze. Secondary derived functors and the Adams spectral sequence. Topology. 45(2006) 295-324.
- [10] T. Pirashvili. Projective and injective objects in symmetric categorical groups. arxiv:hep-th/1007.0121v1.
- [11] T. Pirashvili. On abelian 2-categories and derived 2-functors. arxiv:hep-th/1007.0121v1.

Fang Huang, Shao-Han Chen, Wei Chen
 Department of Mathematics
 South China University of Technology
 Guangzhou 510641, P. R. China

Zhu-Jun Zheng
 Department of Mathematics
 South China University of Technology
 Guangzhou 510641, P. R. China
 and
 Institute of Mathematics
 Henan University
 Kaifeng 475001, P. R. China
 E-mail: zhengzj@scut.edu.cn